IX вјежбе

Парцијалне диференцијалне једначине

1. Нека је Ω област у $\mathbb{R}^n$ и нека су ф-је u и v хармонијске на Ω. Доказати да важи:

a) Ф-ја $\varphi(x) = \langle x, \operatorname{grad} u(x) \rangle$ је хармонијска на Ω.

б) Ф-ја $u \cdot v$ је хармонијска на Ω ако је $\langle \operatorname{grad} u, \operatorname{grad} v \rangle \equiv 0$ на Ω.

Рјешење:

а) u је хармонијска на $\Omega \Rightarrow \Delta u = 0$ на Ω.

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right)$$

$$x = (x_1, x_2, \dots, x_n)$$

$$\varphi(x) = \langle x, \operatorname{grad} u(x) \rangle = \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x)$$

$$\frac{\partial \varphi}{\partial x_j}(x) = \frac{\partial u}{\partial x_j}(x) + \sum_{i=1}^{n} x_i \frac{\partial^2 u}{\partial x_i \partial x_j}(x), \quad j = 1, \dots, n$$

$$\frac{\partial^2 \varphi}{\partial x_j^2}(x) = 2 \frac{\partial^2 u}{\partial x_j^2}(x) + \sum_{i=1}^{n} x_i \frac{\partial^3 u}{\partial x_i \partial x_j^2}(x), \quad j = 1, \dots, n$$

$$\Delta \varphi = \sum_{j=1}^{n} \frac{\partial^2 \varphi}{\partial x_j^2} = 2 \cdot \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{n} \sum_{i=1}^{n} x_i \frac{\partial^3 u}{\partial x_i \partial x_j^2} =$$

$$= 2 \cdot \Delta u + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \right) = 2 \cdot 0 + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} (\Delta u) =$$

$$= 2 \cdot 0 + \sum_{i=1}^{n} x_i \cdot \frac{\partial}{\partial x_i}(0) = 0 \text{ на } \Omega \Rightarrow$$

$\Rightarrow \varphi(x)$ је хармонијска на Ω

б) $u\cdot\vartheta$ - хармонијска на $\Omega \Leftrightarrow \Delta(u\cdot\vartheta)\equiv 0$ на Ω

$$\frac{\partial(u\cdot\vartheta)}{\partial x_i} = \frac{\partial u}{\partial x_i}\cdot\vartheta + u\frac{\partial\vartheta}{\partial x_i}$$

$$\frac{\partial^2(u\cdot\vartheta)}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\cdot\vartheta + u\cdot\frac{\partial\vartheta}{\partial x_i}\right) = \frac{\partial^2 u}{\partial x_i^2}\cdot\vartheta + \frac{\partial u}{\partial x_i}\frac{\partial\vartheta}{\partial x_i} + \frac{\partial u}{\partial x_i}\cdot\frac{\partial\vartheta}{\partial x_i} +$$

$$+ u\cdot\frac{\partial^2\vartheta}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2}\cdot\vartheta + 2\frac{\partial u}{\partial x_i}\frac{\partial\vartheta}{\partial x_i} + u\cdot\frac{\partial^2\vartheta}{\partial x_i^2}$$

$$\Delta(u\cdot\vartheta) = \sum_{i=1}^{n}\frac{\partial^2(u\cdot\vartheta)}{\partial x_i^2} = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i}\cdot\vartheta + 2\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial\vartheta}{\partial x_i} +$$

$$+\sum_{i=1}^{n} u\cdot\frac{\partial^2\vartheta}{\partial x_i} = \vartheta\cdot\Delta u + 2\cdot\langle \mathrm{grad}\,u, \mathrm{grad}\,\vartheta\rangle + u\cdot\Delta\vartheta$$

Како су u и ϑ хармонијске на Ω, што је
$\Delta(u\cdot\vartheta) = 2\cdot\langle \mathrm{grad}\,u, \mathrm{grad}\,\vartheta\rangle$ на Ω.
Дакле $\Delta(u\cdot\vartheta)\equiv 0$ на Ω акко је $\langle \mathrm{grad}\,u, \mathrm{grad}\,\vartheta\rangle \equiv 0$ на Ω.

2. Нека је ф-ја $u:\mathbb{R}^2\to\mathbb{R}$ хармонијска на $\Omega\subseteq\mathbb{R}^2$.
Да ли су следеће ф-је хармонијске на Ω:

а) $\vartheta(x,y) = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$

б) $\vartheta(x,y) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$

в) $\vartheta(x,y) = y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y}$

Рјешење:

а) $$\frac{\partial\vartheta}{\partial x} = \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y\partial x}$$

$$\frac{\partial\vartheta}{\partial y} = \frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 \vartheta}{\partial x^2} = \frac{\partial^3 u}{\partial x^3}\frac{\partial u}{\partial y} + 2\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y \partial x^2}$$

$$\frac{\partial^2 \vartheta}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}\frac{\partial u}{\partial y} + 2\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial y^3}$$

$$\Delta \vartheta = \frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} = \frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\frac{\partial^2 u}{\partial x \partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) +$$

$$+ \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial x}(\Delta u) + 2\frac{\partial^2 u}{\partial x \partial y}\cdot\Delta u +$$

$$+ \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial y}\cdot(\Delta u) \equiv 0 \text{ на } \Omega \Rightarrow$$

$\Rightarrow \vartheta$ је хармонијска на Ω.

д) $$\frac{\partial \vartheta}{\partial x} = 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial \vartheta}{\partial y} = 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 \vartheta}{\partial x^2} = 2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x^2 \partial y}$$

$$\frac{\partial^2 \vartheta}{\partial y^2} = 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x \partial y^2} + 2\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^3}$$

$$\Delta \vartheta = \frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} = 2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 4\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + 2\left(\frac{\partial^2 u}{\partial y^2}\right)^2 +$$

$$+ 2\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) =$$

$$= 2\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2\right) + 2\cdot\frac{\partial u}{\partial x}\frac{\partial}{\partial x}(\Delta u) +$$

$$+ 2\frac{\partial u}{\partial y}\frac{\partial}{\partial y}(\Delta u) = 2\left(\left(\frac{\partial^2 u}{\partial x^2}\right) + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right) + \left(\frac{\partial^2 u}{\partial y^2}\right)^2\right) \text{ на } \Omega$$

Изаберимо $u(x,y)=x\cdot y$. Очигледно, u је хармонијска на $\mathbb{R}^2$, па је хармонијска и на $\Omega \subseteq \mathbb{R}^2$. Тада је

$$\Delta \vartheta(x,y) = 2\cdot\left(0^2+2\cdot 1^2+0^2\right) = 2 \neq 0 \text{ на } \Omega.$$

Дакле, ф-ја $\vartheta = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$ није хармонијска ф-ја у општем случају.

в)
$$\frac{\partial \vartheta}{\partial x} = y\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} - x\frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial \vartheta}{\partial y} = \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} - x\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 \vartheta}{\partial x^2} = y\frac{\partial^3 u}{\partial x^3} - 2\frac{\partial^2 u}{\partial x\partial y} - x\frac{\partial^3 u}{\partial x^2\partial y}$$

$$\frac{\partial^2 \vartheta}{\partial y^2} = 2\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^3 u}{\partial x\partial y^2} - x\frac{\partial^3 u}{\partial y^3}$$

$$\Delta\vartheta = \frac{\partial^2\vartheta}{\partial x^2} + \frac{\partial^2\vartheta}{\partial y^2} = y\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) - x\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) +$$
$$+ 2\cancel{\frac{\partial^2 u}{\partial x\partial y}} - 2\cancel{\frac{\partial^2 u}{\partial x\partial y}} = y\cdot\frac{\partial}{\partial x}(\Delta u) - x\frac{\partial}{\partial y}(\Delta u) \equiv 0 \text{ на } \Omega \Rightarrow$$

$\Rightarrow \vartheta$ је хармонијска на Ω

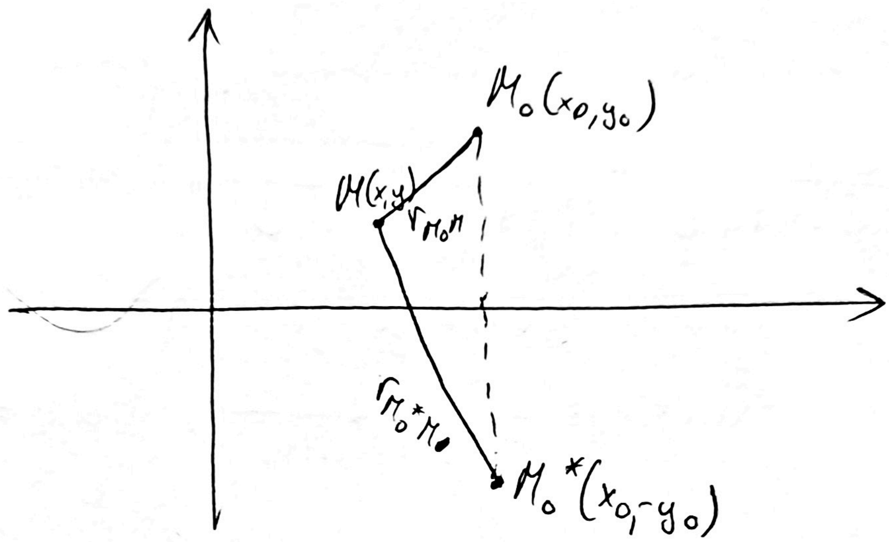
3. Одредити рјешење Дирихлеовог задатка

$$\Delta u = 0$$

$$u(x,0) = f(x) = \begin{cases} 1, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$$

на полуравни $\{(x,y) \mid y>0\}$.

Рјешење:



$M_0^*(x_0,-y_0)$ - тачка симетрична тачки M_0 у односу на границу области (Ox-осу)

Знамо да је $G(x,y) = E(x,y) + g(x,y)$, гдје је

$$E(x,y) = E(x,y,x_0,y_0) = \frac{1}{2\pi} \ln r_{M_0M} = \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2+(y-y_0)^2}$$

фундаментално рјешење.

Даље је

$$g(x,y) = g(x,y,x_0,y_0) = -\frac{1}{2\pi} \ln r_{M_0^*M} = -\frac{1}{2\pi} \ln \sqrt{(x-x_0)^2+(y+y_0)^2}$$

па је

$$G(x,y,x_0,y_0) = \frac{1}{2\pi}\left(\ln \sqrt{(x-x_0)^2+(y-y_0)^2} - \ln \sqrt{(x-x_0)^2+(y+y_0)^2} \right) =$$

$$= \frac{1}{4\pi} \ln \frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}$$

Примијетимо да је за $M \in Ox$

$$G(x,y,x_0,y_0) = \frac{1}{4\pi} \ln \frac{(x-x_0)^2 + y_0^2}{(x-x_0)^2 + (y_0)^2} = 0$$

Даље је

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y} = -\frac{1}{4\pi} \left(\frac{2(y-y_0)}{(x-x_0)^2 + (y-y_0)^2} - \frac{2(y+y_0)}{(x-x_0)^2 + (y+y_0)^2} \right)$$

* Извод по спољњем вектору нормале на границу области

а $$\left.\frac{\partial G}{\partial n}\right|_{y_0=0} = -\frac{1}{4\pi} \left(\frac{-2y}{(x-x_0)^2 + (y-0)^2} - \frac{2y}{(x-x)^2 + (y+0)^2} \right) = \frac{y}{\pi((x-x_0)^2 + y^2)}$$

Дакле, рјешење задатка је

$$u(x,y) = \frac{1}{\pi} \int \frac{y}{(x-x_0)^2 + y_0^2} \cdot f(x_0)\, dx_0 =$$

$$= \frac{y}{\pi} \int_a^b \frac{1}{(x-x_0)^2 + y^2}\, dx_0 = \frac{y}{\pi} \cdot \frac{1}{y^2} \int_a^b \frac{1}{\left(\frac{x-x_0}{y}\right)^2 + 1}\, dx_0 =$$

$$= -\frac{1}{\pi} \cdot \operatorname{arctg} \frac{x-x_0}{y} \Big|_a^b = -\frac{1}{\pi} \left(\operatorname{arctg} \frac{x-b}{y} - \operatorname{arctg} \frac{x-a}{y} \right)$$